Reframing Control as an Inference Problem

CS 294-112: Deep Reinforcement Learning

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Class Notes

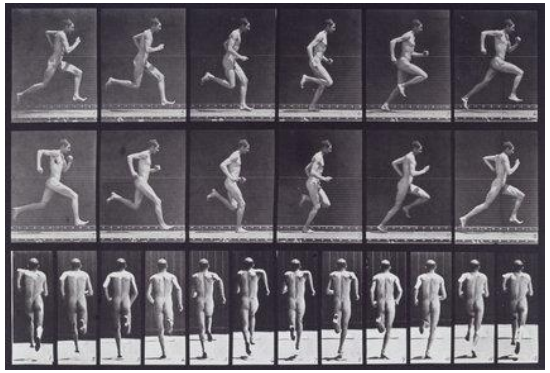
1. Homework 3 is due today, at 11:59 pm
2. Peer reviews will be out shortly!
3. Homework 4 comes out tonight

Today's Lecture

1. Does reinforcement learning and optimal control provide a reasonable model of human behavior?
2. Is there a better explanation?
3. Can we derive optimal control, reinforcement learning, and planning as *probabilistic inference*?
4. How does this change our RL algorithms?
5. (next week) We'll see this is crucial for *inverse* reinforcement learning

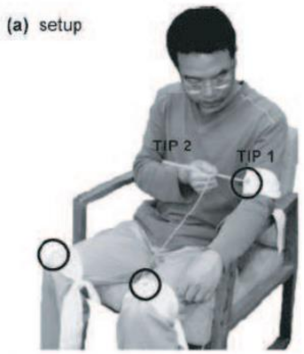
- Goals:
  - Understand the connection between inference and control
  - Understand how specific RL algorithms can be instantiated in this framework
  - Understand why this might be a good idea

Optimal Control as a Model of Human Behavior



Muybridge (c. 1870)

Mombaur et al. ‘09



Li & Todorov ‘06

Ziebart ‘08

$$\mathbf{a}_1, \ldots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \ldots, \mathbf{a}_T} \sum_{t=1}^{T} r(\mathbf{s}_t, \mathbf{a}_t)$$

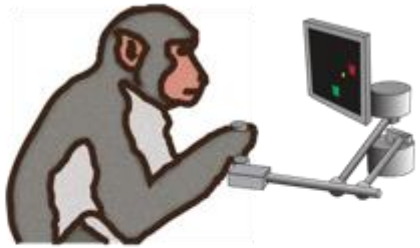
$$\mathbf{s}_{t+1} = f(\mathbf{s}_t, \mathbf{a}_t)$$

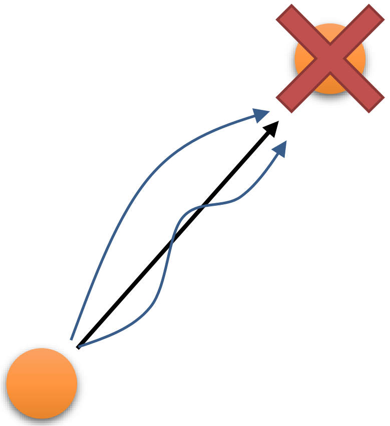
$$\pi = \arg \max_{\pi} E_{\mathbf{s}_{t+1} \sim p(\mathbf{s}_{t+1} | \mathbf{s}_t, \mathbf{a}_t), \mathbf{a}_t \sim \pi(\mathbf{a}_t | \mathbf{s}_t)} [r(\mathbf{s}_t, \mathbf{a}_t)]$$

$$\mathbf{a}_t \sim \pi(\mathbf{a}_t | \mathbf{s}_t)$$

optimize this to explain the data

What if the data is **not** optimal?





some mistakes matter more than others!

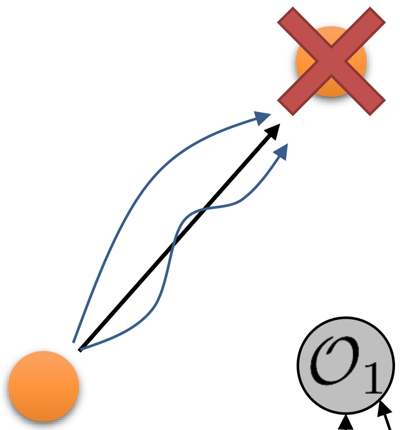
behavior is **stochastic**

but good behavior is still the most likely

A probabilistic graphical model of decision making

$$\mathbf{a}_1, \ldots, \mathbf{a}_T = \arg \max_{\mathbf{a}_1, \ldots, \mathbf{a}_T} \sum_{t=1}^{T} r(\mathbf{s}_t, \mathbf{a}_t)$$

$$\mathbf{s}_{t+1} = f(\mathbf{s}_t, \mathbf{a}_t)$$



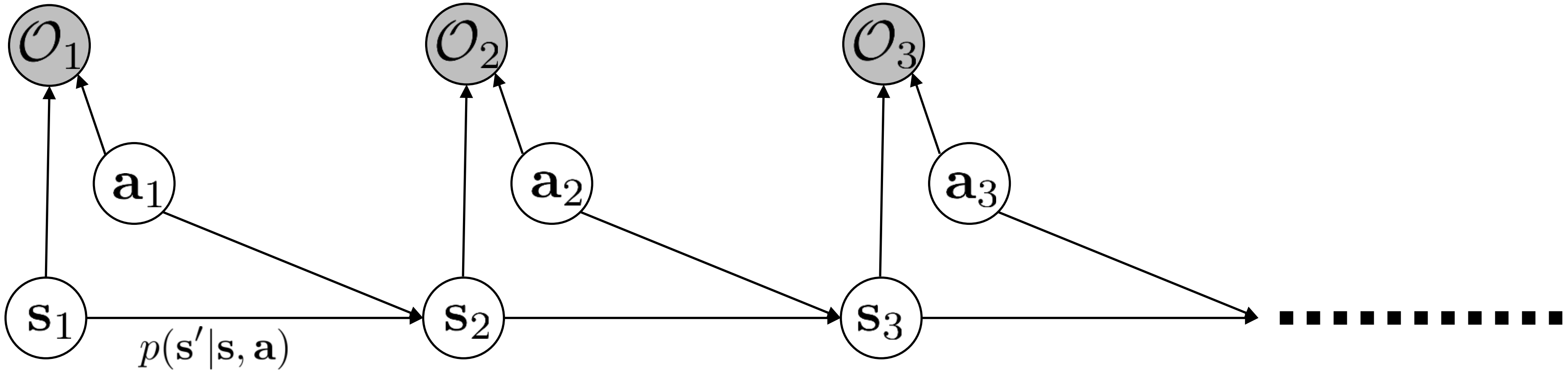
$$\underbrace{p(\mathbf{s}_{1:T}, \mathbf{a}_{1:T})}_{\tau} = ??$$

no assumption of optimal behavior!

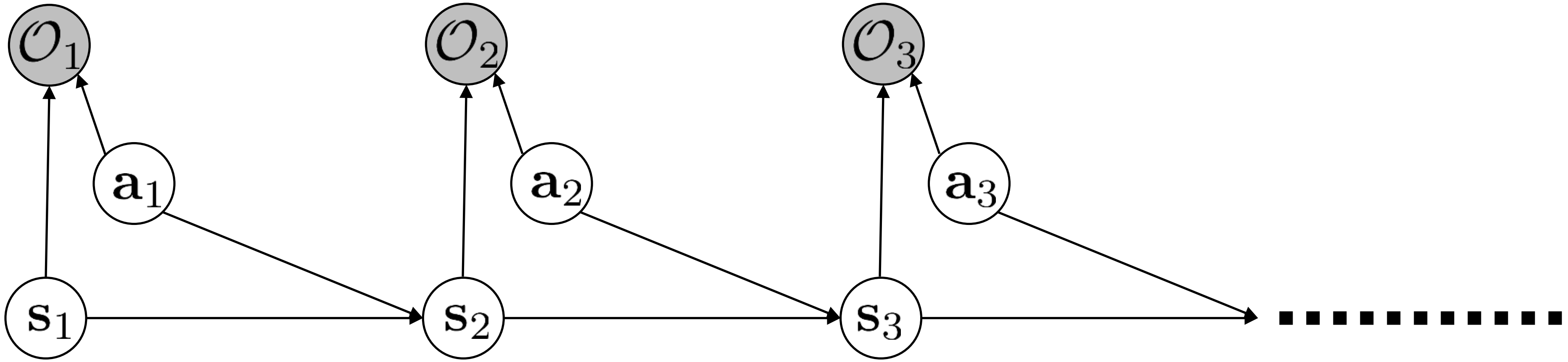
$$p(\tau | \mathcal{O}_{1:T}) \qquad p(\mathcal{O}_t | \mathbf{s}_t, \mathbf{a}_t) \propto \exp(r(\mathbf{s}_t, \mathbf{a}_t))$$

$$p(\tau | \mathcal{O}_{1:T}) = \frac{p(\tau, \mathcal{O}_{1:T})}{p(\mathcal{O}_{1:T})}$$

$$\propto p(\tau) \prod_t \exp(r(\mathbf{s}_t, \mathbf{a}_t)) = p(\tau) \exp\left(\sum_t r(\mathbf{s}_t, \mathbf{a}_t)\right)$$



Why is this interesting?

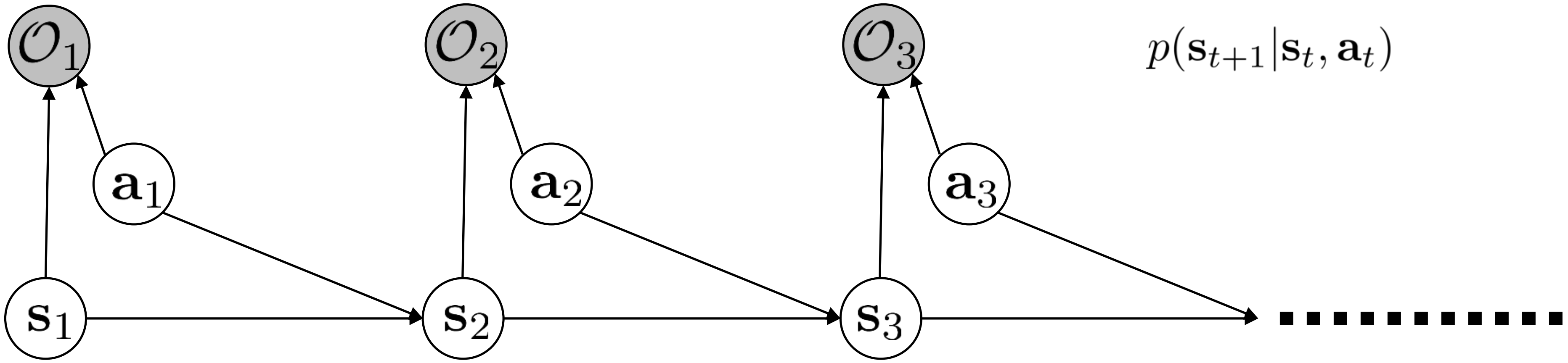


- Can model suboptimal behavior (important for inverse RL)
- Can apply inference algorithms to solve control and planning problems
- Provides an explanation for why stochastic behavior might be preferred (useful for exploration and transfer learning)

Inference = planning

$p(\mathcal{O}_t|\mathbf{s}_t, \mathbf{a}_t) \propto \exp(r(\mathbf{s}_t, \mathbf{a}_t))$

$p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$



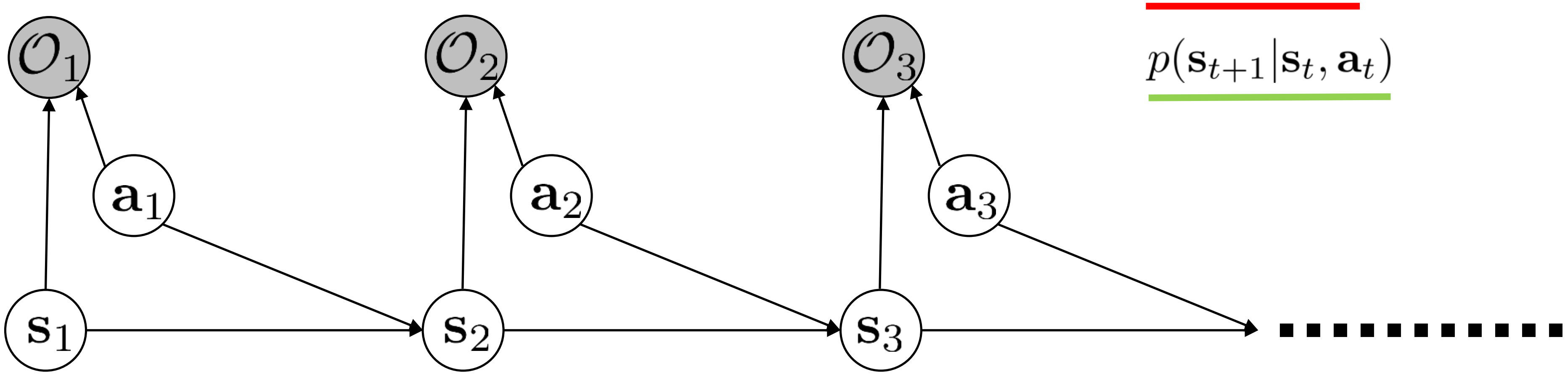
how to do inference?

1. compute backward messages $\beta_t(\mathbf{s}_t, \mathbf{a}_t) = p(\mathcal{O}_{t:T}|\mathbf{s}_t, \mathbf{a}_t)$
2. compute policy $p(\mathbf{a}_t|\mathbf{s}_t, \mathcal{O}_{1:T})$
3. compute forward messages $\alpha_t(\mathbf{s}_t) = p(\mathbf{s}_t|\mathcal{O}_{1:t-1})$

Backward messages

$$p(\mathcal{O}_t|\mathbf{s}_t,\mathbf{a}_t) \propto \exp(r(\mathbf{s}_t,\mathbf{a}_t))$$

$$p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t)$$



$$\beta_t(\mathbf{s}_t,\mathbf{a}_t) = p(\mathcal{O}_{t:T}|\mathbf{s}_t,\mathbf{a}_t)$$
$$= \int p(\mathcal{O}_{t:T},\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t)d\mathbf{s}_{t+1}$$
$$= \int p(\mathcal{O}_{t+1:T}|\mathbf{s}_{t+1})p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t)p(\mathcal{O}_t|\mathbf{s}_t,\mathbf{a}_t)d\mathbf{s}_{t+1}$$

$$p(\mathcal{O}_{t+1:T}|\mathbf{s}_{t+1}) = \int p(\mathcal{O}_{t+1:T}|\mathbf{s}_{t+1},\mathbf{a}_{t+1})p(\mathbf{a}_{t+1}|\mathbf{s}_{t+1})d\mathbf{a}_{t+1}$$

$\beta_t(\mathbf{s}_{t+1},\mathbf{a}_{t+1})$

which actions are likely *a priori*
(assume uniform for now)

for $t = T-1$ to 1:

$$\beta_t(\mathbf{s}_t,\mathbf{a}_t) = p(\mathcal{O}_t|\mathbf{s}_t,\mathbf{a}_t)E_{\mathbf{s}_{t+1}\sim p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t)}[\beta_{t+1}(\mathbf{s}_{t+1})]$$

$$\beta_t(\mathbf{s}_t) = E_{\mathbf{a}_t\sim p(\mathbf{a}_t|\mathbf{s}_t)}[\beta_t(\mathbf{s}_t,\mathbf{a}_t)]$$

A closer look at the backward pass

value iteration algorithm:

1. set $Q(\mathbf{s}, \mathbf{a}) \leftarrow r(\mathbf{s}, \mathbf{a}) + \gamma E[V(\mathbf{s}')]$
2. set $V(\mathbf{s}) \leftarrow \max_{\mathbf{a}} Q(\mathbf{s}, \mathbf{a})$

for $t = T-1$ to 1:

$$\beta_t(\mathbf{s}_t, \mathbf{a}_t) = p(\mathcal{O}_t|\mathbf{s}_t, \mathbf{a}_t) E_{\mathbf{s}_{t+1} \sim p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)}[\beta_{t+1}(\mathbf{s}_{t+1})]$$

$$\beta_t(\mathbf{s}_t) = E_{\mathbf{a}_t \sim p(\mathbf{a}_t|\mathbf{s}_t)}[\beta_t(\mathbf{s}_t, \mathbf{a}_t)]$$

"optimistic" transition (not a good idea!)

$$Q_t(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + \underbrace{\log E[\exp(V_{t+1}(\mathbf{s}_{t+1}))]}$$

let $V_t(\mathbf{s}_t) = \log \beta_t(\mathbf{s}_t)$

let $Q_t(\mathbf{s}_t, \mathbf{a}_t) = \log \beta_t(\mathbf{s}_t, \mathbf{a}_t)$

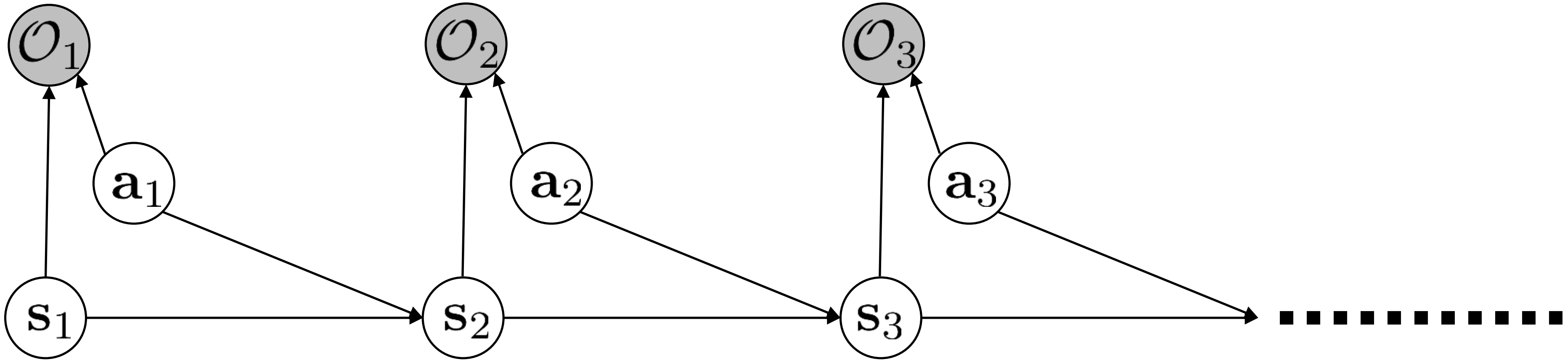
deterministic transition: $Q_t(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + V_{t+1}(\mathbf{s}_{t+1})$

we'll come back to the stochastic case later!

$$V_t(\mathbf{s}_t) = \log \int \exp(Q_t(\mathbf{s}_t, \mathbf{a}_t)) d\mathbf{a}_t$$

$V_t(\mathbf{s}_t) \rightarrow \max_{\mathbf{a}_t} Q_t(\mathbf{s}_t, \mathbf{a}_t)$ as $Q_t(\mathbf{s}_t, \mathbf{a}_t)$ gets bigger!

Backward pass summary



$$\beta_t(\mathbf{s}_t, \mathbf{a}_t) = p(\mathcal{O}_{t:T}|\mathbf{s}_t, \mathbf{a}_t)$$

probability that we can be optimal at steps t through T given that we take action $\mathbf{a}_t$ in state $\mathbf{s}_t$

for $t = T - 1$ to 1:

$$\beta_t(\mathbf{s}_t, \mathbf{a}_t) = p(\mathcal{O}_t|\mathbf{s}_t, \mathbf{a}_t) E_{\mathbf{s}_{t+1} \sim p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)}[\beta_{t+1}(\mathbf{s}_{t+1})]$$

compute recursively from $t = T$ to $t = 1$

$$\beta_t(\mathbf{s}_t) = E_{\mathbf{a}_t \sim p(\mathbf{a}_t|\mathbf{s}_t)}[\beta_t(\mathbf{s}_t, \mathbf{a}_t)]$$

let $V_t(\mathbf{s}_t) = \log \beta_t(\mathbf{s}_t)$

let $Q_t(\mathbf{s}_t, \mathbf{a}_t) = \log \beta_t(\mathbf{s}_t, \mathbf{a}_t)$

log of β_t is "Q-function-like"

The action prior

remember this?

$$p(\mathcal{O}_{t+1:T}|\mathbf{s}_{t+1}) = \int p(\mathcal{O}_{t+1:T}|\mathbf{s}_{t+1}, \mathbf{a}_{t+1})p(\mathbf{a}_{t+1}|\mathbf{s}_{t+1})d\mathbf{a}_{t+1}$$

$\beta_t(\mathbf{s}_{t+1}, \mathbf{a}_{t+1})$

what if the action prior is not uniform?

("soft max")

$$V(\mathbf{s}_t) = \log \int \exp(Q(\mathbf{s}_t, \mathbf{a}_t) + \log p(\mathbf{a}_t|\mathbf{s}_t))\mathbf{a}_t$$

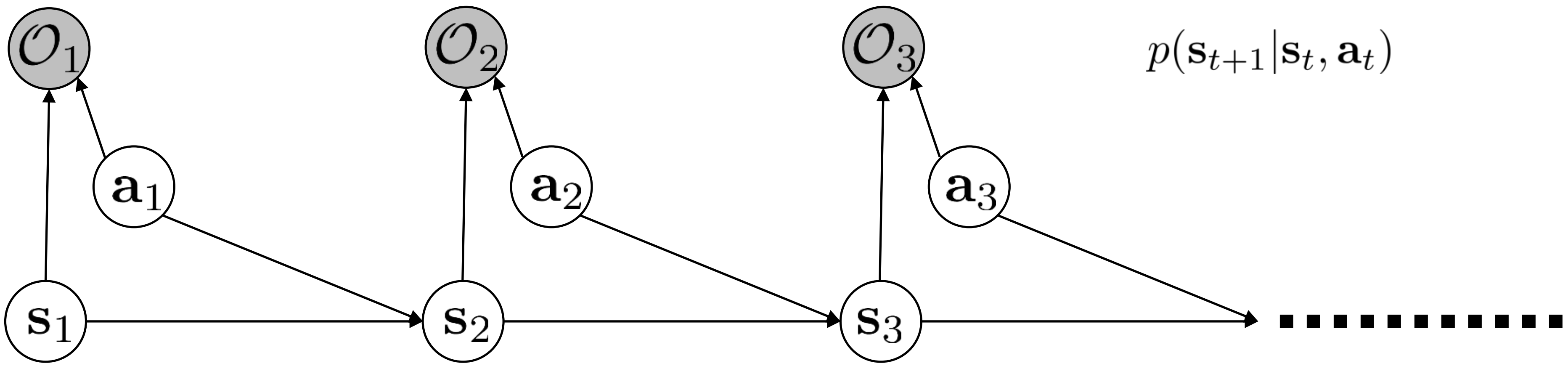
$$Q(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + \log E[\exp(V(\mathbf{s}_{t+1}))]$$

let $\tilde{Q}(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + \log p(\mathbf{a}_t|\mathbf{s}_t) + \log E[\exp(V(\mathbf{s}_{t+1}))]$

$$V(\mathbf{s}_t) = \log \int \exp(\tilde{Q}(\mathbf{s}_t, \mathbf{a}_t))\mathbf{a}_t \quad \Leftrightarrow \quad V(\mathbf{s}_t) = \log \int \exp(Q(\mathbf{s}_t, \mathbf{a}_t) + \log p(\mathbf{a}_t|\mathbf{s}_t))\mathbf{a}_t$$

can **always** fold the action prior into the reward! uniform action prior can be assumed without loss of generality

Policy computation



$p(\mathcal{O}_t|\mathbf{s}_t, \mathbf{a}_t) \propto \exp(r(\mathbf{s}_t, \mathbf{a}_t))$

$p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$

2. compute policy $p(\mathbf{a}_t|\mathbf{s}_t, \mathcal{O}_{1:T})$

$\beta_t(\mathbf{s}_t, \mathbf{a}_t) = p(\mathcal{O}_{t:T}|\mathbf{s}_t, \mathbf{a}_t)$

$\beta_t(\mathbf{s}_t) = p(\mathcal{O}_{t:T}|\mathbf{s}_t)$

$$\begin{aligned}
p(\mathbf{a}_t|\mathbf{s}_t, \mathcal{O}_{1:T}) &= \pi(\mathbf{a}_t|\mathbf{s}_t) \\
&= p(\mathbf{a}_t|\mathbf{s}_t, \mathcal{O}_{t:T}) \\
&= \frac{p(\mathbf{a}_t, \mathbf{s}_t|\mathcal{O}_{t:T})}{p(\mathbf{s}_t|\mathcal{O}_{t:T})} \\
&= \frac{p(\mathcal{O}_{t:T}|\mathbf{a}_t, \mathbf{s}_t)p(\mathbf{a}_t, \mathbf{s}_t)/\cancel{p(\mathcal{O}_{t:T})}}{p(\mathcal{O}_{t:T}|\mathbf{s}_t)p(\mathbf{s}_t)/\cancel{p(\mathcal{O}_{t:T})}} \\
&= \frac{p(\mathcal{O}_{t:T}|\mathbf{a}_t, \mathbf{s}_t)}{p(\mathcal{O}_{t:T}|\mathbf{s}_t)} \frac{p(\mathbf{a}_t, \mathbf{s}_t)}{p(\mathbf{s}_t)} = \frac{\beta_t(\mathbf{s}_t, \mathbf{a}_t)}{\beta_t(\mathbf{s}_t)} \cancel{p(\mathbf{a}_t|\mathbf{s}_t)}
\end{aligned}$$

$$\pi(\mathbf{a}_t|\mathbf{s}_t) = \frac{\beta_t(\mathbf{s}_t, \mathbf{a}_t)}{\beta_t(\mathbf{s}_t)}$$

Policy computation with value functions

for $t = T - 1$ to 1:

$$Q_t(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + \log E[\exp(V_{t+1}(\mathbf{s}_{t+1}))]$$

$$V_t(\mathbf{s}_t) = \log \int \exp(Q_t(\mathbf{s}_t, \mathbf{a}_t))\mathbf{a}_t$$

$$\pi(\mathbf{a}_t|\mathbf{s}_t) = \frac{\beta_t(\mathbf{s}_t, \mathbf{a}_t)}{\beta_t(\mathbf{s}_t)}$$

$$V_t(\mathbf{s}_t) = \log \beta_t(\mathbf{s}_t)$$

$$Q_t(\mathbf{s}_t, \mathbf{a}_t) = \log \beta_t(\mathbf{s}_t, \mathbf{a}_t)$$

$$\pi(\mathbf{a}_t|\mathbf{s}_t) = \exp(Q_t(\mathbf{s}_t, \mathbf{a}_t) - V_t(\mathbf{s}_t)) = \exp(A_t(\mathbf{s}_t, \mathbf{a}_t))$$

Policy computation summary

$$\pi(\mathbf{a}_t|\mathbf{s}_t) = \exp(Q_t(\mathbf{s}_t, \mathbf{a}_t) - V_t(\mathbf{s}_t)) = \exp(A_t(\mathbf{s}_t, \mathbf{a}_t))$$

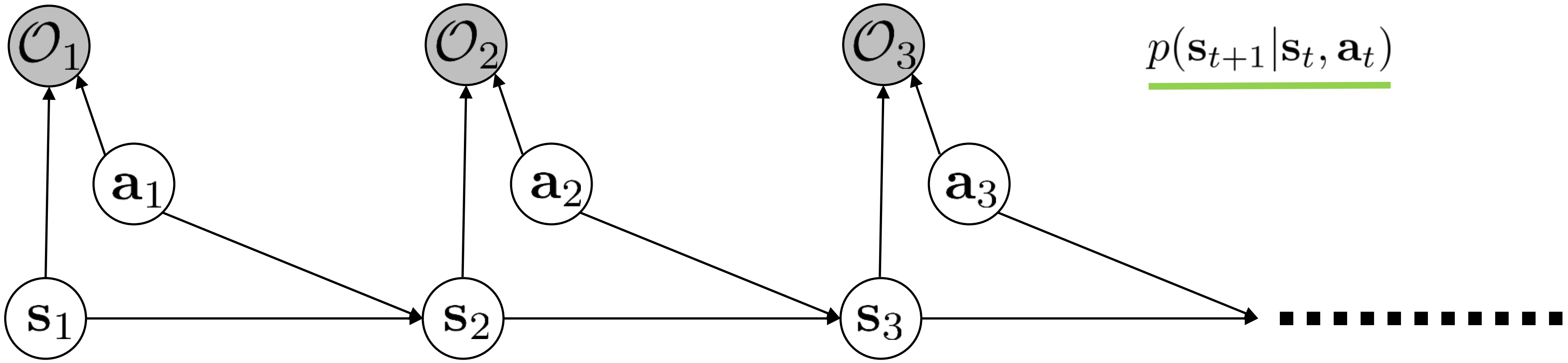
$$\text{with temperature: } \pi(\mathbf{a}_t|\mathbf{s}_t) = \exp(\tfrac{1}{\alpha}Q_t(\mathbf{s}_t, \mathbf{a}_t) - \tfrac{1}{\alpha}V_t(\mathbf{s}_t)) = \exp(\tfrac{1}{\alpha}A_t(\mathbf{s}_t, \mathbf{a}_t))$$

- Natural interpretation: better actions are more probable
- Random tie-breaking
- Analogous to Boltzmann exploration
- Approaches greedy policy as temperature decreases

Forward messages

$$p(\mathcal{O}_t|\mathbf{s}_t,\mathbf{a}_t) \propto \exp(r(\mathbf{s}_t,\mathbf{a}_t))$$

$$\underline{p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t)}$$



$$\alpha_t(\mathbf{s}_t) = p(\mathbf{s}_t|\mathcal{O}_{1:t-1})$$

$$= \int \underline{p(\mathbf{s}_t|\mathbf{s}_{t-1},\mathbf{a}_{t-1})} p(\mathbf{a}_{t-1}|\mathbf{s}_{t-1},\mathcal{O}_{t-1}) \underbrace{p(\mathbf{s}_{t-1}|\mathcal{O}_{1:t-2})}_{\alpha_{t-1}(\mathbf{s}_{t-1})} d\mathbf{s}_{t-1} d\mathbf{a}_{t-1}$$

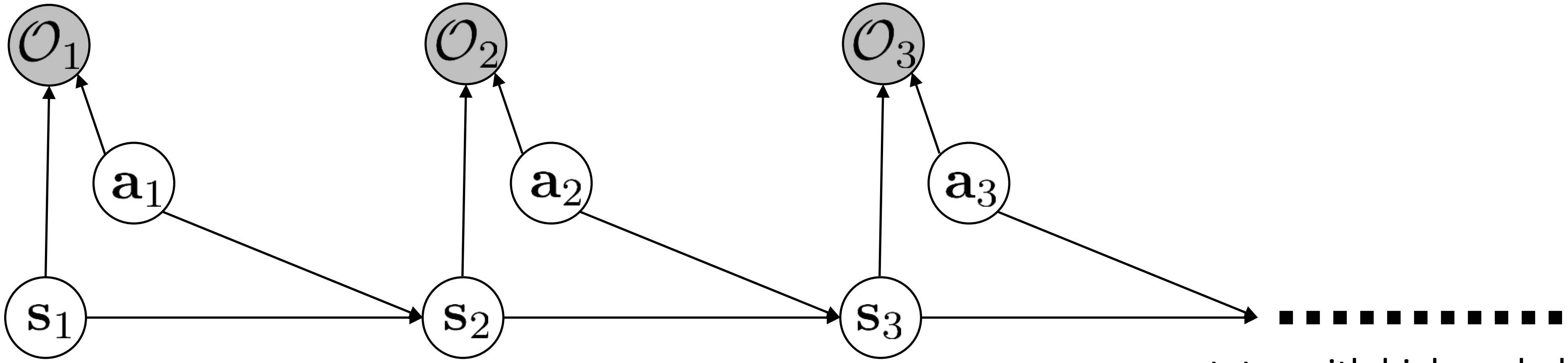
$$\alpha_1(\mathbf{s}_1) = p(\mathbf{s}_1) \text{ (usually known)}$$

$$p(\mathbf{a}_{t-1}|\mathbf{s}_{t-1},\mathcal{O}_{t-1}) = \frac{p(\mathcal{O}_{t-1}|\mathbf{s}_{t-1},\mathbf{a}_{t-1})p(\mathbf{a}_{t-1}|\mathbf{s}_{t-1})}{p(\mathcal{O}_{t-1}|\mathbf{s}_{t-1})}$$

what if we want $p(\mathbf{s}_t|\mathcal{O}_{1:T})$?

$$p(\mathbf{s}_t|\mathcal{O}_{1:T}) = \frac{p(\mathbf{s}_t,\mathcal{O}_{1:T})}{p(\mathcal{O}_{1:T})} = \frac{\overbrace{p(\mathcal{O}_{t:T}|\mathbf{s}_t)}^{\beta_t(\mathbf{s}_t)}p(\mathbf{s}_t,\mathcal{O}_{1:t-1})}{p(\mathcal{O}_{1:T})} \propto \beta_t(\mathbf{s}_t)\underbrace{p(\mathbf{s}_t|\mathcal{O}_{1:t-1})}_{\alpha_t(\mathbf{s}_t)}\cancel{p(\mathcal{O}_{1:t-1})} \propto \beta_t(\mathbf{s}_t)\alpha_t(\mathbf{s}_t)$$

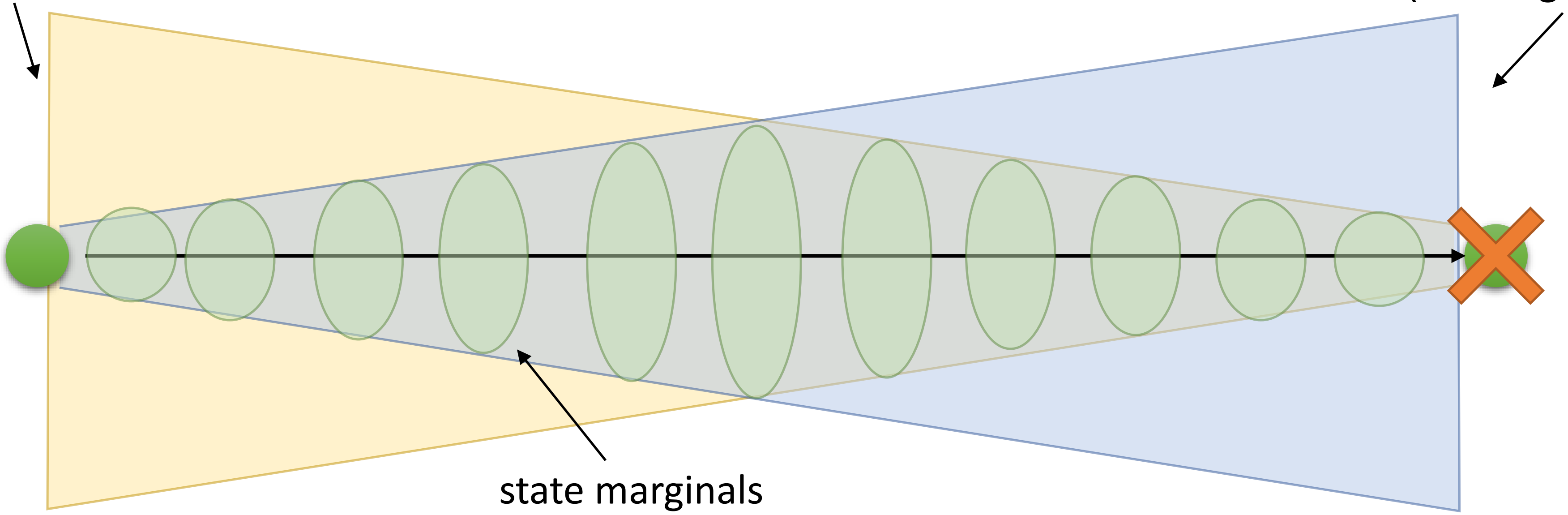
Forward/backward message intersection



states with high probability of reaching goal

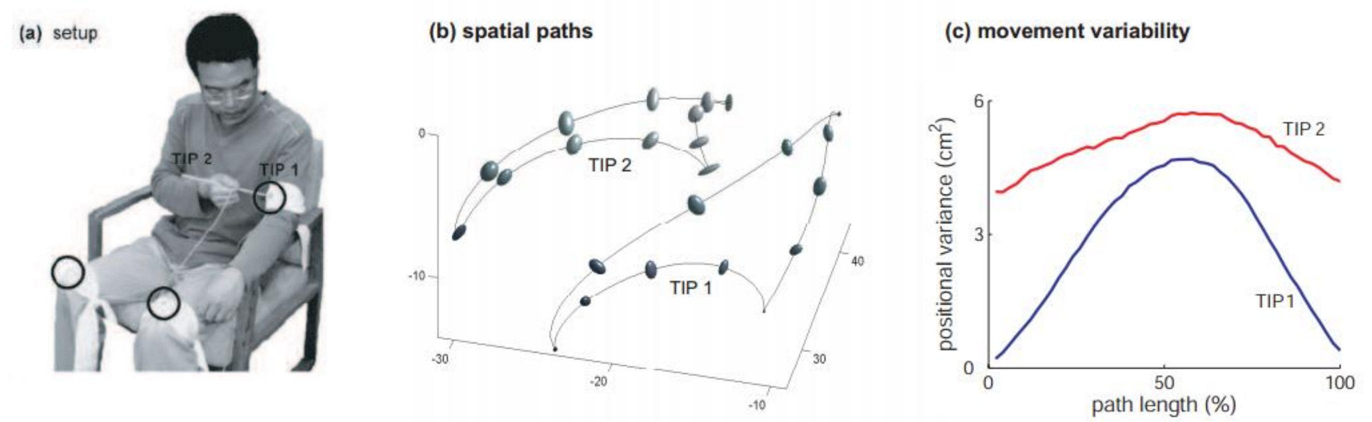
$$p(\mathbf{s}_t) \propto \beta_t(\mathbf{s}_t)\alpha_t(\mathbf{s}_t)$$

states with high probability of being reached from initial state (with high reward)



state marginals

Forward/backward message intersection

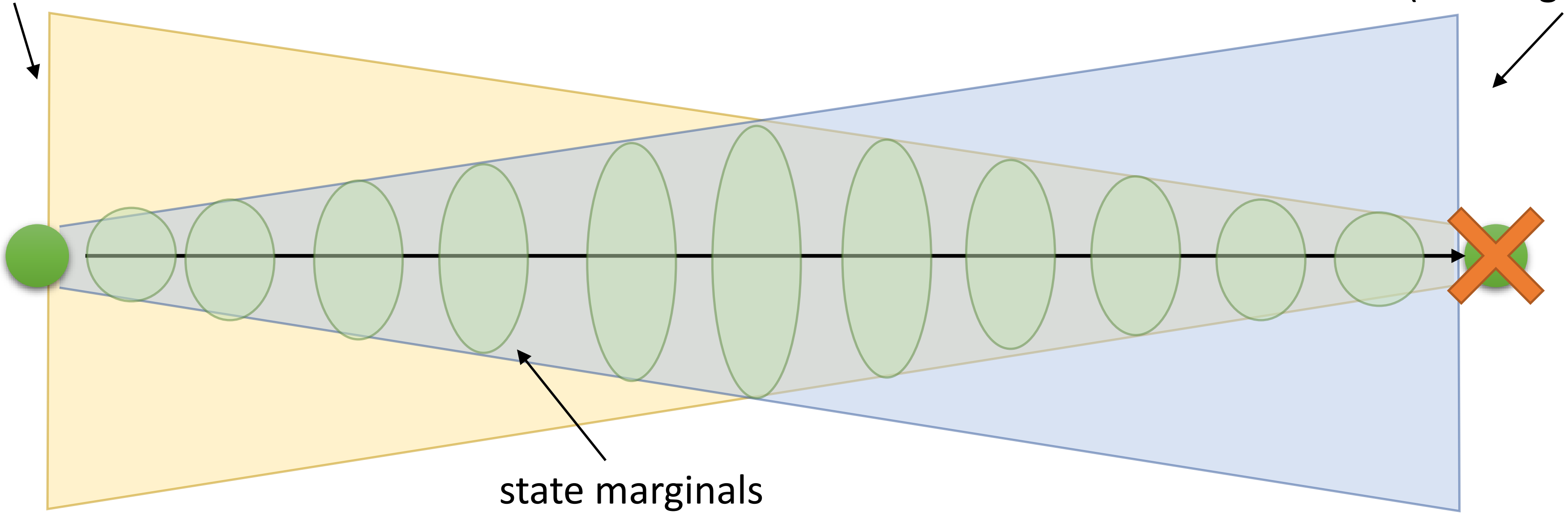


Li & Todorov, 2006

states with high probability of reaching goal

$$p(\mathbf{s}_t) \propto \beta_t(\mathbf{s}_t)\alpha_t(\mathbf{s}_t)$$

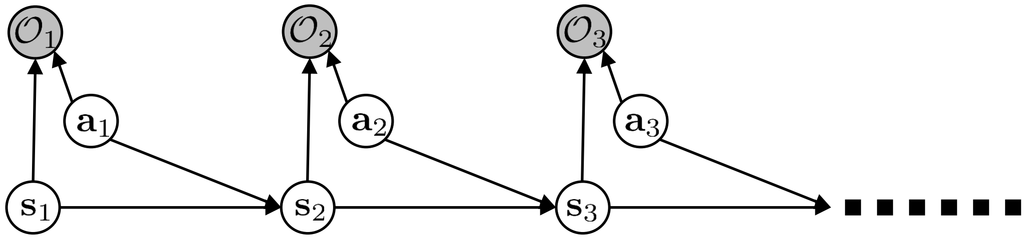
states with high probability of being reached from initial state (with high reward)



state marginals

Summary

1. Probabilistic graphical model for optimal control



2. Control = inference (similar to HMM, EKF, etc.)

3. Very similar to dynamic programming, value iteration, etc. (but “soft”)

Break

The optimism problem

for $t = T-1$ to 1:

$$\beta_t(\mathbf{s}_t, \mathbf{a}_t) = p(\mathcal{O}_t|\mathbf{s}_t, \mathbf{a}_t) E_{\mathbf{s}_{t+1} \sim p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)}[\beta_{t+1}(\mathbf{s}_{t+1})]$$

$$\beta_t(\mathbf{s}_t) = E_{\mathbf{a}_t \sim p(\mathbf{a}_t|\mathbf{s}_t)}[\beta_t(\mathbf{s}_t, \mathbf{a}_t)]$$

let $V_t(\mathbf{s}_t) = \log \beta_t(\mathbf{s}_t)$

let $Q_t(\mathbf{s}_t, \mathbf{a}_t) = \log \beta_t(\mathbf{s}_t, \mathbf{a}_t)$

"optimistic" transition
(not a good idea!)

$$Q_t(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + \overbrace{\log E[\exp(V_{t+1}(\mathbf{s}_{t+1}))]}$$

why did this happen?

the inference problem: $p(\mathbf{s}_{1:T}, \mathbf{a}_{1:T}|\mathcal{O}_{1:T})$

marginalizing and conditioning, we get: $p(\mathbf{a}_t|\mathbf{s}_t, \mathcal{O}_{1:T})$ (the policy)

"given that you obtained high reward, what was your action probability?"

marginalizing and conditioning, we get: $p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t, \mathcal{O}_{1:T}) \neq p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$

"given that you obtained high reward, what was your transition probability?"

Addressing the optimism problem

marginalizing and conditioning, we get: $p(\mathbf{a}_t|\mathbf{s}_t, \mathcal{O}_{1:T})$ (the policy) ← we want this

"given that you obtained high reward, what was your action probability?"

marginalizing and conditioning, we get: $p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t, \mathcal{O}_{1:T}) \neq p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$ ← but not this!

"given that you obtained high reward, what was your transition probability?"

"given that you obtained high reward, what was your action probability,

given that your transition probability did not change?"

can we find another distribution $q(\mathbf{s}_{1:T}, \mathbf{a}_{1:T})$ that is close to $p(\mathbf{s}_{1:T}, \mathbf{a}_{1:T}|\mathcal{O}_{1:T})$ but has dynamics $p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t)$

where have we seen this before? let $\mathbf{x} = \mathcal{O}_{1:T}$ and $\mathbf{z} = (\mathbf{s}_{1:T}, \mathbf{a}_{1:T})$ find $q(\mathbf{z})$ to approximate $p(\mathbf{z}|\mathbf{x})$

let's try variational inference!

Control via variational inference

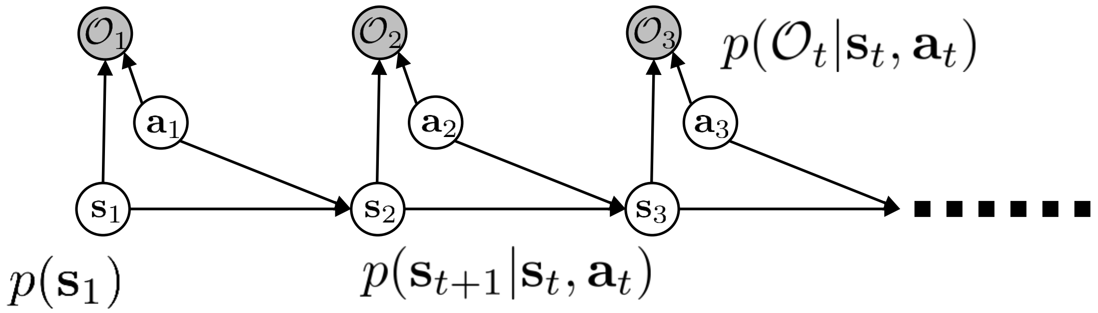
let $q(\mathbf{s}_{1:T}, \mathbf{a}_{1:T}) = p(\mathbf{s}_1) \prod_t p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t) q(\mathbf{a}_t|\mathbf{s}_t)$

same dynamics and initial state as p

only new thing

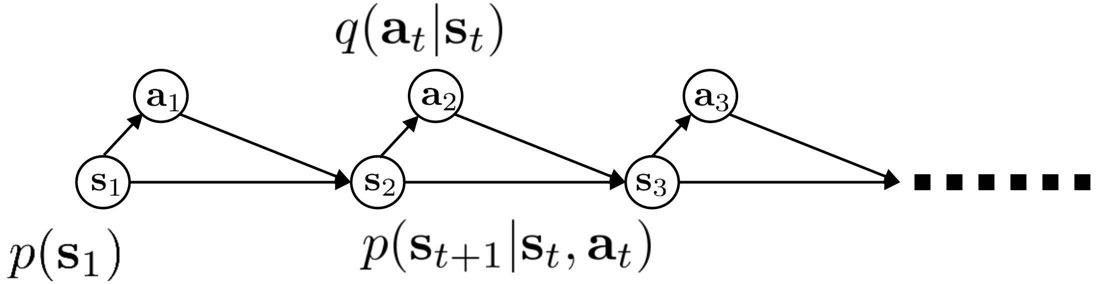
let $\mathbf{x} = \mathcal{O}_{1:T}$ and $\mathbf{z} = (\mathbf{s}_{1:T}, \mathbf{a}_{1:T})$

$p(\mathbf{s}_{1:T}, \mathbf{a}_{1:T}|\mathcal{O}_{1:T})$



$p(\mathbf{z}|\mathbf{x})$

$q(\mathbf{s}_{1:T}, \mathbf{a}_{1:T})$



$q(\mathbf{z})$

The variational lower bound

$$\log p(\mathbf{x}) \geq E_{\mathbf{z}\sim q(\mathbf{z})}[\log p(\mathbf{x},\mathbf{z}) - \log q(\mathbf{z})]$$

let $\mathbf{x} = \mathcal{O}_{1:T}$ and $\mathbf{z} = (\mathbf{s}_{1:T}, \mathbf{a}_{1:T})$

the entropy $\mathcal{H}(q)$

let $q(\mathbf{s}_{1:T}, \mathbf{a}_{1:T}) = p(\mathbf{s}_1) \prod_t p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t) q(\mathbf{a}_t|\mathbf{s}_t)$

$$\log p(\mathcal{O}_{1:T}) \geq E_{(\mathbf{s}_{1:T},\mathbf{a}_{1:T})\sim q}\Big[\log p(\mathbf{s}_1) + \sum_{t=1}^T \log p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t) + \sum_{t=1}^T \log p(\mathcal{O}_t|\mathbf{s}_t,\mathbf{a}_t)$$

$$- \log p(\mathbf{s}_1) - \sum_{t=1}^T \log p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t) - \sum_{t=1}^T \log q(\mathbf{a}_t|\mathbf{s}_t)\Big]$$

$$= E_{(\mathbf{s}_{1:T},\mathbf{a}_{1:T})\sim q}\left[\sum_t r(\mathbf{s}_t,\mathbf{a}_t) - \log q(\mathbf{a}_t|\mathbf{s}_t)\right]$$

$$= \sum_t E_{(\mathbf{s}_t,\mathbf{a}_t)\sim q}[r(\mathbf{s}_t,\mathbf{a}_t) + \mathcal{H}(q(\mathbf{a}_t|\mathbf{s}_t))]$$

← maximize reward and maximize action entropy!

Optimizing the variational lower bound

let $q(\mathbf{s}_{1:T}, \mathbf{a}_{1:T}) = p(\mathbf{s}_1) \prod_t p(\mathbf{s}_{t+1}|\mathbf{s}_t, \mathbf{a}_t) q(\mathbf{a}_t|\mathbf{s}_t)$

$$\log p(\mathcal{O}_{1:T}) \geq \sum_t E_{(\mathbf{s}_t, \mathbf{a}_t) \sim q} [r(\mathbf{s}_t, \mathbf{a}_t) + \mathcal{H}(q(\mathbf{a}_t|\mathbf{s}_t))]$$

base case: solve for $q(\mathbf{a}_T|\mathbf{s}_T)$:

$$q(\mathbf{a}_T|\mathbf{s}_T) = \arg\max E_{\mathbf{s}_T \sim q(\mathbf{s}_T)} \left[E_{\mathbf{a}_T \sim q(\mathbf{a}_T|\mathbf{s}_T)} [r(\mathbf{s}_T, \mathbf{a}_T)] + \mathcal{H}(q(\mathbf{a}_T|\mathbf{s}_T)) \right]$$

$$= \arg\max E_{\mathbf{s}_T \sim q(\mathbf{s}_T)} \left[E_{\mathbf{a}_T \sim q(\mathbf{a}_T|\mathbf{s}_T)} [r(\mathbf{s}_T, \mathbf{a}_T) - \log q(\mathbf{a}_T|\mathbf{s}_T)] \right]$$

minimized when $q(\mathbf{a}_T|\mathbf{s}_T) \propto \exp(r(\mathbf{s}_T, \mathbf{a}_T))$

$$q(\mathbf{a}_T|\mathbf{s}_T) = \frac{\exp(r(\mathbf{s}_T, \mathbf{a}_T))}{\int \exp(r(\mathbf{s}_T, \mathbf{a})) d\mathbf{a}} = \exp(Q(\mathbf{s}_T, \mathbf{a}_T) - V(\mathbf{s}_T)) \qquad V(\mathbf{s}_T) = \log \int \exp(Q(\mathbf{s}_T, \mathbf{a}_T)) d\mathbf{a}_T$$

$$E_{\mathbf{s}_T \sim q(\mathbf{s}_T)} \left[E_{\mathbf{a}_T \sim q(\mathbf{a}_T|\mathbf{s}_T)} [r(\mathbf{s}_T, \mathbf{a}_T) - \log q(\mathbf{a}_T|\mathbf{s}_T)] \right] = E_{\mathbf{s}_T \sim q(\mathbf{s}_T)} \left[E_{\mathbf{a}_T \sim q(\mathbf{a}_T|\mathbf{s}_T)} [V(\mathbf{s}_T)] \right]$$

Optimizing the variational lower bound

$$\log p(\mathcal{O}_{1:T}) \geq \sum_t E_{(\mathbf{s}_t,\mathbf{a}_t)\sim q}\left[r(\mathbf{s}_t,\mathbf{a}_t) + \mathcal{H}(q(\mathbf{a}_t|\mathbf{s}_t))\right]$$

$$q(\mathbf{a}_T|\mathbf{s}_T) = \frac{\exp(r(\mathbf{s}_T,\mathbf{a}_T))}{\int \exp(r(\mathbf{s}_T,\mathbf{a}))d\mathbf{a}} = \exp(Q(\mathbf{s}_T,\mathbf{a}_T) - V(\mathbf{s}_T))$$

$$E_{\mathbf{s}_T\sim q(\mathbf{s}_T)}\left[E_{\mathbf{a}_T\sim q(\mathbf{a}_T|\mathbf{s}_T)}[r(\mathbf{s}_T,\mathbf{a}_T) - \log q(\mathbf{a}_T|\mathbf{s}_T)]\right] = E_{\mathbf{s}_T\sim q(\mathbf{s}_T)}\left[E_{\mathbf{a}_T\sim q(\mathbf{a}_T|\mathbf{s}_T)}[V(\mathbf{s}_T)]\right]$$

$$\begin{aligned} q(\mathbf{a}_t|\mathbf{s}_t) &= \arg\max E_{\mathbf{s}_t\sim q(\mathbf{s}_t)}\left[E_{\mathbf{a}_t\sim q(\mathbf{a}_t|\mathbf{s}_t)}[r(\mathbf{s}_t,\mathbf{a}_t) + E_{\mathbf{s}_{t+1}\sim p(\mathbf{s}_{t+1}|\mathbf{s}_t,\mathbf{a}_t)}[V(\mathbf{s}_{t+1})]] + \mathcal{H}(q(\mathbf{a}_t|\mathbf{s}_t))\right] \\ &= \arg\max E_{\mathbf{s}_t\sim q(\mathbf{s}_t)}\left[E_{\mathbf{a}_t\sim q(\mathbf{a}_t|\mathbf{s}_t)}[Q(\mathbf{s}_t,\mathbf{a}_t)] + \mathcal{H}(q(\mathbf{a}_t|\mathbf{s}_t))\right] \\ &= \arg\max E_{\mathbf{s}_t\sim q(\mathbf{s}_t)}\left[E_{\mathbf{a}_t\sim q(\mathbf{a}_t|\mathbf{s}_t)}[Q(\mathbf{s}_t,\mathbf{a}_t) - \log q(\mathbf{a}_t|\mathbf{s}_t)]\right] \end{aligned}$$

minimized when $q(\mathbf{a}_t|\mathbf{s}_t) \propto \exp(Q(\mathbf{s}_t,\mathbf{a}_t))$

$$V_t(\mathbf{s}_t) = \log \int \exp(Q_t(\mathbf{s}_t,\mathbf{a}_t))d\mathbf{a}_t$$

$$q(\mathbf{a}_t|\mathbf{s}_t) = \exp(Q(\mathbf{s}_t,\mathbf{a}_t) - V(\mathbf{s}_t))$$

$$Q_t(\mathbf{s}_t,\mathbf{a}_t) = r(\mathbf{s}_t,\mathbf{a}_t) + E[(V_{t+1}(\mathbf{s}_{t+1})]$$

↑ *regular* Bellman backup
not optimistic

Backward pass summary - variational

for $t = T - 1$ to 1:

$$Q_t(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + E[(V_{t+1}(\mathbf{s}_{t+1})]$$

$$V_t(\mathbf{s}_t) = \log \int \exp(Q_t(\mathbf{s}_t, \mathbf{a}_t)) d\mathbf{a}_t$$

value iteration algorithm:

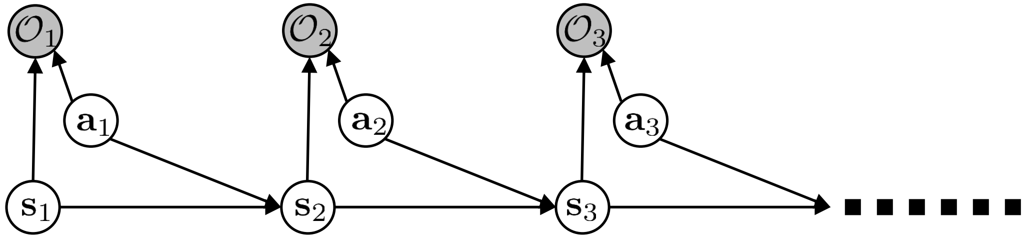
1. set $Q(\mathbf{s}, \mathbf{a}) \leftarrow r(\mathbf{s}, \mathbf{a}) + \gamma E[V(\mathbf{s}')]$
2. set $V(\mathbf{s}) \leftarrow \max_{\mathbf{a}} Q(\mathbf{s}, \mathbf{a})$

soft value iteration algorithm:

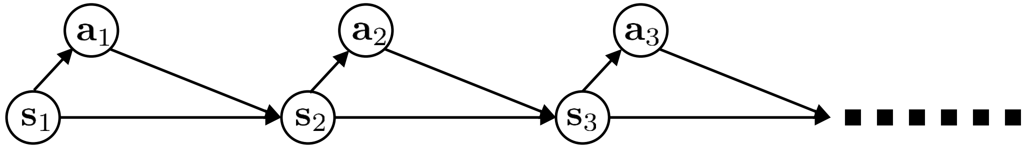
1. set $Q(\mathbf{s}, \mathbf{a}) \leftarrow r(\mathbf{s}, \mathbf{a}) + \gamma E[V(\mathbf{s}')]$
2. set $V(\mathbf{s}) \leftarrow \text{soft}\max_{\mathbf{a}} Q(\mathbf{s}, \mathbf{a})$

Summary

$p(\mathbf{s}_{1:T}, \mathbf{a}_{1:T} | \mathcal{O}_{1:T})$



$q(\mathbf{s}_{1:T}, \mathbf{a}_{1:T})$



$$V_t(\mathbf{s}_t) = \log \int \exp(Q_t(\mathbf{s}_t, \mathbf{a}_t)) d\mathbf{a}_t \qquad Q_t(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + E[(V_{t+1}(\mathbf{s}_{t+1})]$$

variants:

discounted SOC: $Q_t(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + \gamma E[V_{t+1}(\mathbf{s}_{t+1})]$

explicit temperature: $V_t(\mathbf{s}_t) = \alpha \log \int \exp\left(\frac{1}{\alpha} Q_t(\mathbf{s}_t, \mathbf{a}_t)\right) d\mathbf{a}_t$

For more details, see: Levine. (2018). Reinforcement Learning and Control as Probabilistic Inference: Tutorial and Review.

Q-learning with soft optimality

standard Q-learning: $\phi \leftarrow \phi + \alpha \nabla_\phi Q_\phi(\mathbf{s},\mathbf{a})(r(\mathbf{s},\mathbf{a}) + \gamma V(\mathbf{s}') - Q_\phi(\mathbf{s},\mathbf{a}))$

target value: $V(\mathbf{s}') = \max_{\mathbf{a}'} Q_\phi(\mathbf{s}',\mathbf{a}')$

soft Q-learning: $\phi \leftarrow \phi + \alpha \nabla_\phi Q_\phi(\mathbf{s},\mathbf{a})(r(\mathbf{s},\mathbf{a}) + \gamma V(\mathbf{s}') - Q_\phi(\mathbf{s},\mathbf{a}))$

target value: $V(\mathbf{s}') = \text{soft}\max_{\mathbf{a}'} Q_\phi(\mathbf{s}',\mathbf{a}') = \log \int \exp(Q_\phi(\mathbf{s}',\mathbf{a}'))d\mathbf{a}'$

$\pi(\mathbf{a}|\mathbf{s}) = \exp(Q_\phi(\mathbf{s},\mathbf{a}) - V(\mathbf{s})) = \exp(A(\mathbf{s},\mathbf{a}))$

1. take some action $\mathbf{a}_i$ and observe $(\mathbf{s}_i, \mathbf{a}_i, \mathbf{s}'_i, r_i)$, add it to $\mathcal{R}$
2. sample mini-batch $\{\mathbf{s}_j, \mathbf{a}_j, \mathbf{s}'_j, r_j\}$ from $\mathcal{R}$ uniformly
3. compute $y_j = r_j + \gamma \text{soft}\max_{\mathbf{a}'_j} Q_{\phi'}(\mathbf{s}'_j, \mathbf{a}'_j)$ using *target* network $Q_{\phi'}$
4. $\phi \leftarrow \phi - \alpha \sum_j \frac{dQ_\phi}{d\phi}(\mathbf{s}_j, \mathbf{a}_j)(Q_\phi(\mathbf{s}_j, \mathbf{a}_j) - y_j)$
5. update ϕ': copy ϕ every N steps, or Polyak average $\phi' \leftarrow \tau\phi' + (1-\tau)\phi$

Policy gradient with soft optimality

$$\pi(\mathbf{a}|\mathbf{s}) = \exp(Q_\phi(\mathbf{s},\mathbf{a}) - V(\mathbf{s})) \text{ optimizes } \sum_t E_{\pi(\mathbf{s}_t,\mathbf{a}_t)}[r(\mathbf{s}_t,\mathbf{a}_t)] + E_{\pi(\mathbf{s}_t)}[\mathcal{H}(\pi(\mathbf{a}_t|\mathbf{s}_t))]$$

policy entropy

intuition: $\pi(\mathbf{a}|\mathbf{s}) \propto \exp(Q_\phi(\mathbf{s},\mathbf{a}))$ when π minimizes $D_{\mathrm{KL}}(\pi(\mathbf{a}|\mathbf{s}) \| \frac{1}{Z}\exp(Q(\mathbf{s},\mathbf{a})))$

$$D_{\mathrm{KL}}(\pi(\mathbf{a}|\mathbf{s}) \| \tfrac{1}{Z}\exp(Q(\mathbf{s},\mathbf{a}))) = E_{\pi(\mathbf{a}|\mathbf{s})}[Q(\mathbf{s},\mathbf{a})] - \mathcal{H}(\pi)$$

often referred to as "entropy regularized" policy gradient

combats premature entropy collapse

turns out to be closely related to soft Q-learning:
see Haarnoja et al. '17 and Schulman et al. '17

Ziebart et al. '10 "Modeling Interaction via the Principle of Maximum Causal Entropy"

Policy gradient vs Q-learning

policy gradient derivation:

$$J(\theta) = \sum_t E_{\pi(\mathbf{s}_t,\mathbf{a}_t)}[r(\mathbf{s}_t,\mathbf{a}_t)] + E_{\pi(\mathbf{s}_t)}[\underbrace{\mathcal{H}(\pi(\mathbf{a}|\mathbf{s}_t))}_{E_{\pi(\mathbf{a}_t|\mathbf{s}_t)}[-\log \pi(\mathbf{a}_t|\mathbf{s}_t)]}] = \sum_t E_{\pi(\mathbf{s}_t,\mathbf{a}_t)}[r(\mathbf{s}_t,\mathbf{a}_t) - \log \pi(\mathbf{a}_t|\mathbf{s}_t)]$$

$$\nabla_\theta \left[\sum_t E_{\pi(\mathbf{s}_t,\mathbf{a}_t)}[r(\mathbf{s}_t,\mathbf{a}_t) - \underline{\log \pi(\mathbf{a}_t|\mathbf{s}_t)}] \right]$$

$$\approx \frac{1}{N}\sum_i\sum_t \nabla_\theta \log \pi(\mathbf{a}_t|\mathbf{s}_t)\left(r(\mathbf{s}_t,\mathbf{a}_t) + \underbrace{\left(\sum_{t'=t+1}^{T} r(\mathbf{s}_{t'},\mathbf{a}_{t'}) - \log \pi(\mathbf{a}_{t'}|\mathbf{s}_{t'})\right)}_{\approx Q(\mathbf{s}_{t+1},\mathbf{a}_{t+1})} - \log \pi(\mathbf{a}_t|\mathbf{s}_t) - \cancel{1} \right)$$

can ignore (baseline)

recall: $\log \pi(\mathbf{a}_t|\mathbf{s}_t) = Q(\mathbf{s}_t,\mathbf{a}_t) - V(\mathbf{s}_t)$

$$\approx \frac{1}{N}\sum_i\sum_t \underline{(\nabla_\theta Q(\mathbf{a}_t|\mathbf{s}_t) - \nabla_\theta V(\mathbf{s}_t))}\left(r(\mathbf{s}_t,\mathbf{a}_t) + Q(\mathbf{s}_{t+1},\mathbf{a}_{t+1}) - Q(\mathbf{s}_t,\mathbf{a}_t) + \cancel{V(\mathbf{s}_t)}\right)$$

Q-learning: $$-\frac{1}{N}\sum_i\sum_t \underline{\nabla_\theta Q(\mathbf{a}_t|\mathbf{s}_t)}\left(r(\mathbf{s}_t,\mathbf{a}_t) + \underline{\text{soft}\max_{\mathbf{a}_{t+1}} Q(\mathbf{s}_{t+1},\mathbf{a}_{t+1})} - Q(\mathbf{s}_t,\mathbf{a}_t)\right)$$

descent (vs ascent)

off-policy correction

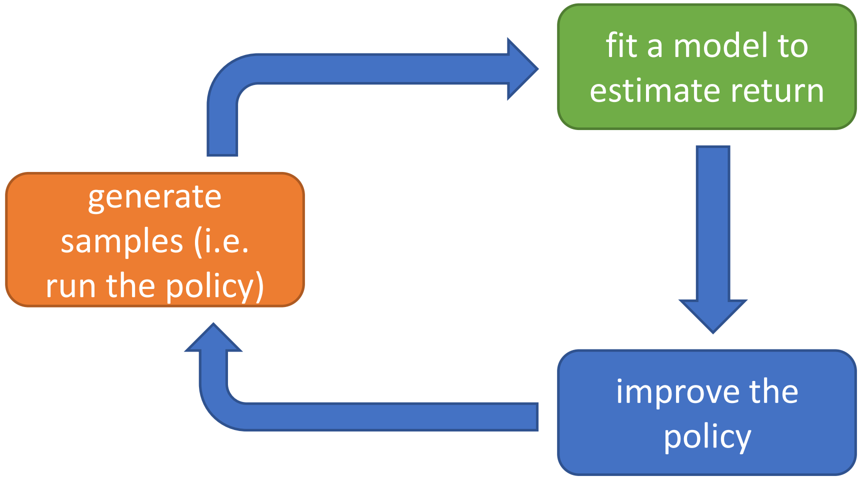
Benefits of soft optimality

- Improve exploration and prevent entropy collapse
- Easier to specialize (finetune) policies for more specific tasks
- Principled approach to break ties
- Better robustness (due to wider coverage of states)
- Can reduce to hard optimality as reward magnitude increases
- Good model for modeling human behavior (more on this later)

Review

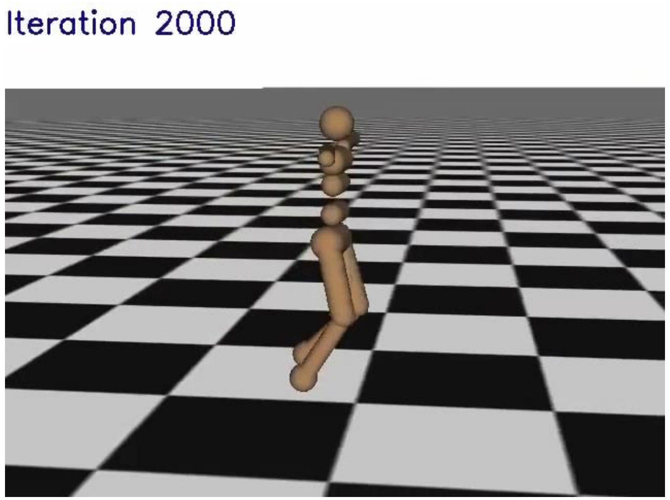
- Reinforcement learning can be viewed as inference in a graphical model
  - Value function is a backward message
  - Maximize reward and entropy (the bigger the rewards, the less entropy matters)
  - Variational inference to remove optimism
- Soft Q-learning
- Entropy-regularized policy gradient

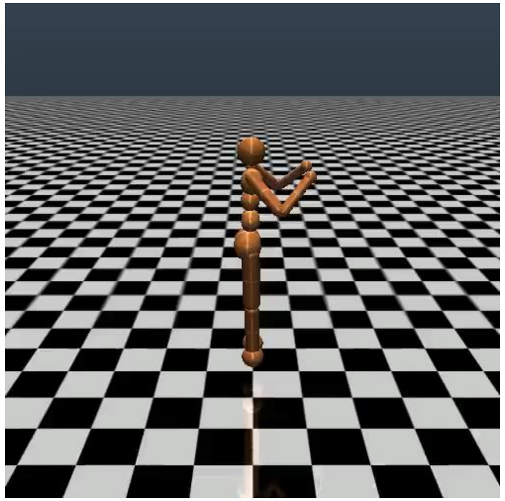
$$Q_\phi(\mathbf{s}, \mathbf{a}) \leftarrow r(\mathbf{s}, \mathbf{a}) + \gamma \text{soft max}_{\mathbf{a}'} Q_\phi(\mathbf{s}', \mathbf{a}')$$

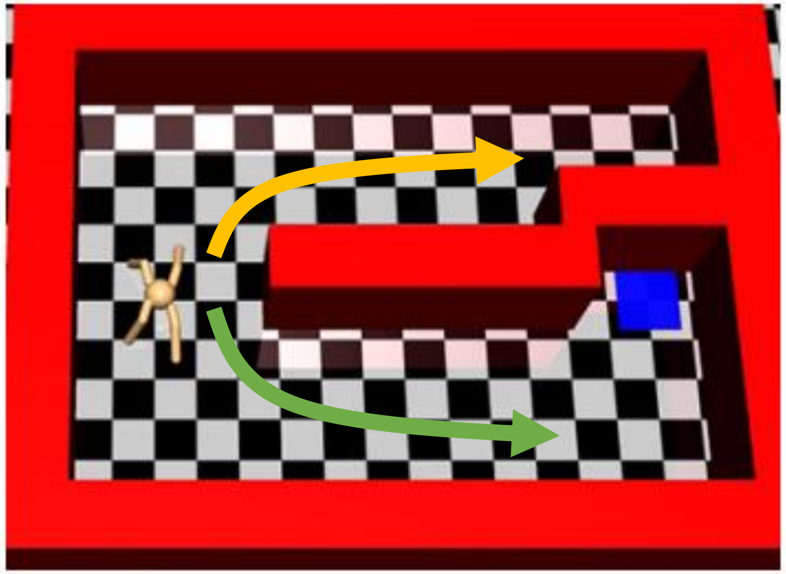


$$\mathbf{a} = \arg\max_{\mathbf{a}} Q_\phi(\mathbf{s}, \mathbf{a})$$

Stochastic models for learning control

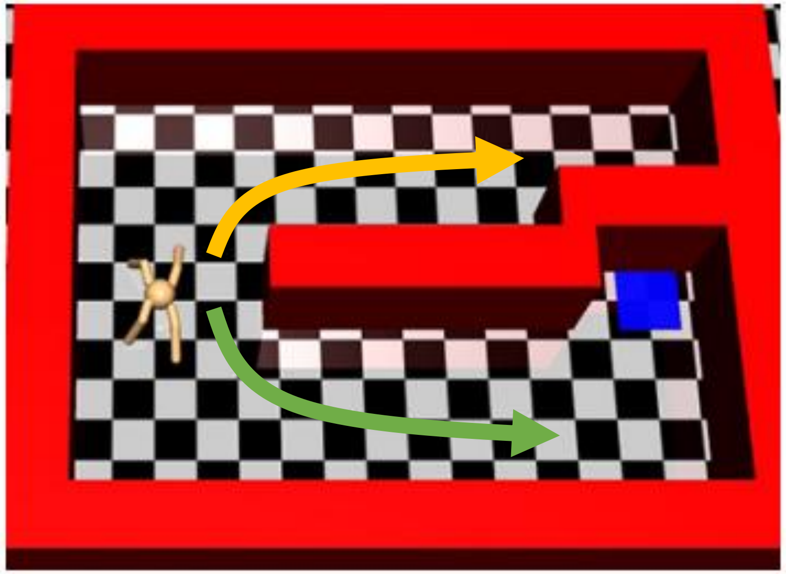




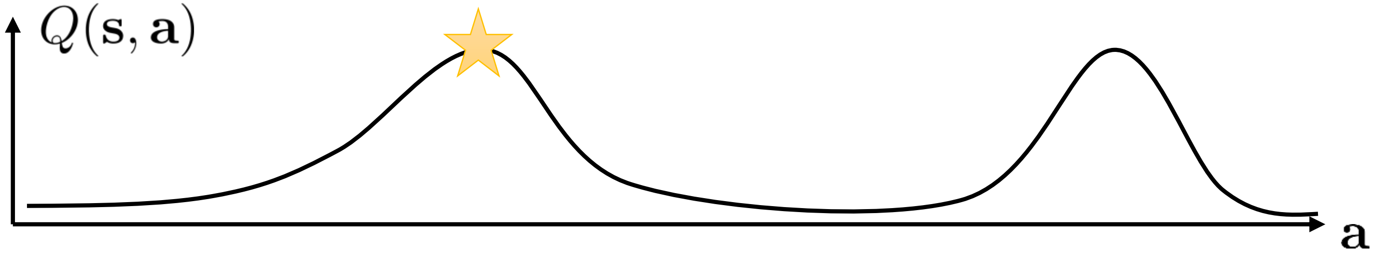


- How can we track *both* hypotheses?

Stochastic energy-based policies



Q-function: $Q(\mathbf{s}, \mathbf{a}) : \mathcal{S} \times \mathcal{A} \to \mathbb{R}$



$$\pi(\mathbf{a}|\mathbf{s}) \propto \exp(Q(\mathbf{s}, \mathbf{a}))$$

$$\pi(\mathbf{a}_t|\mathbf{s}_t) = \exp(Q_t(\mathbf{s}_t, \mathbf{a}_t) - V_t(\mathbf{s}_t)) = \exp(A_t(\mathbf{s}_t, \mathbf{a}_t))$$

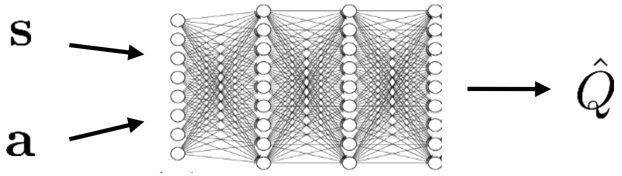
$$Q_t(\mathbf{s}_t, \mathbf{a}_t) = r(\mathbf{s}_t, \mathbf{a}_t) + E[V_{t+1}(\mathbf{s}_{t+1})]$$

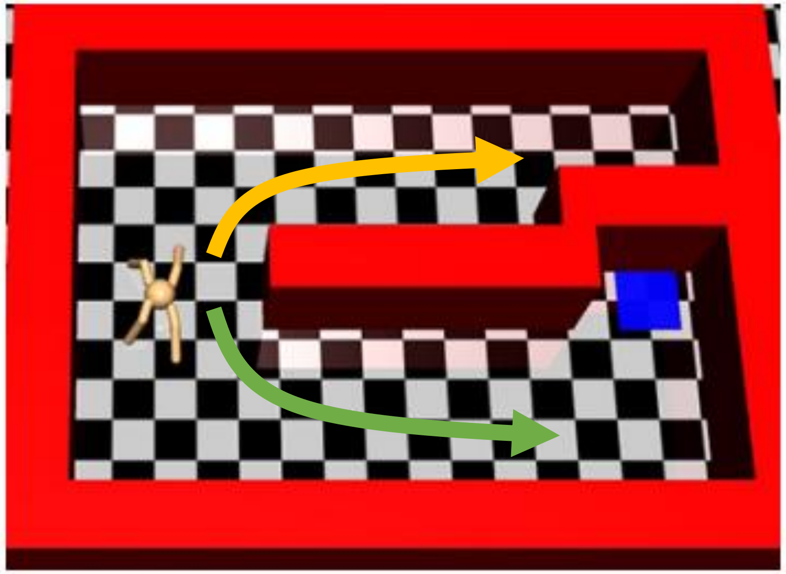
$$V_t(\mathbf{s}_t) = \log \int \exp(Q_t(\mathbf{s}_t, \mathbf{a}_t))\mathbf{a}_t$$

Haarnoja*, Tang*, Abbeel, L., Reinforcement Learning with Deep Energy-Based Policies. ICML 2017

Soft Q-learning

Learned (neural network) Q-function: $Q_\theta(\mathbf{s}, \mathbf{a})$





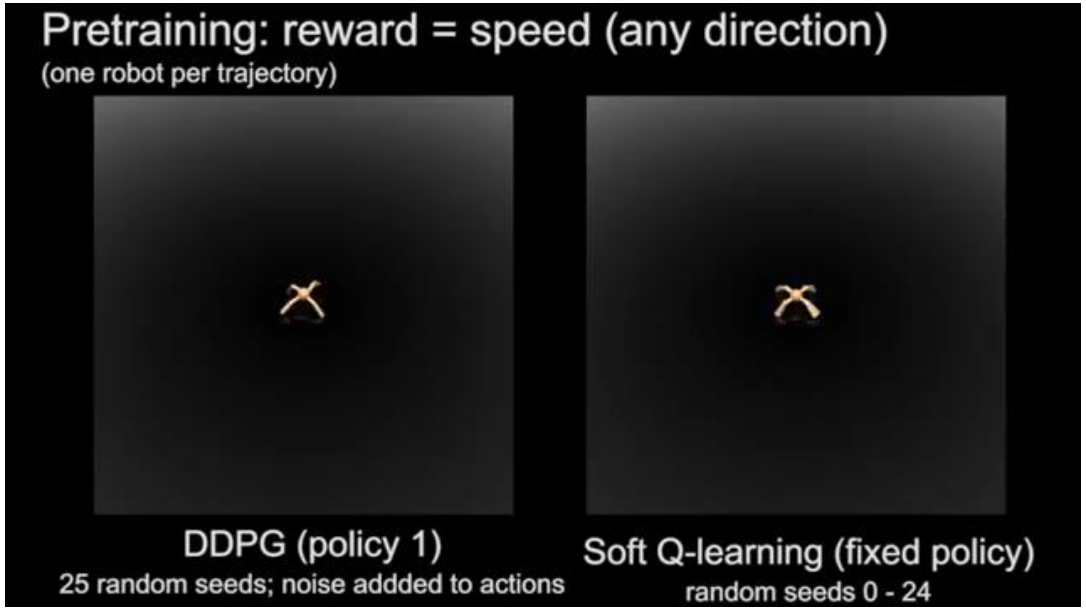
Q-learning: $\theta \leftarrow \theta + \alpha \nabla_\theta Q_\theta(\mathbf{s}, \mathbf{a})(r(\mathbf{s}, \mathbf{a}) + \gamma V(\mathbf{s}') - Q_\theta(\mathbf{s}, \mathbf{a}))$

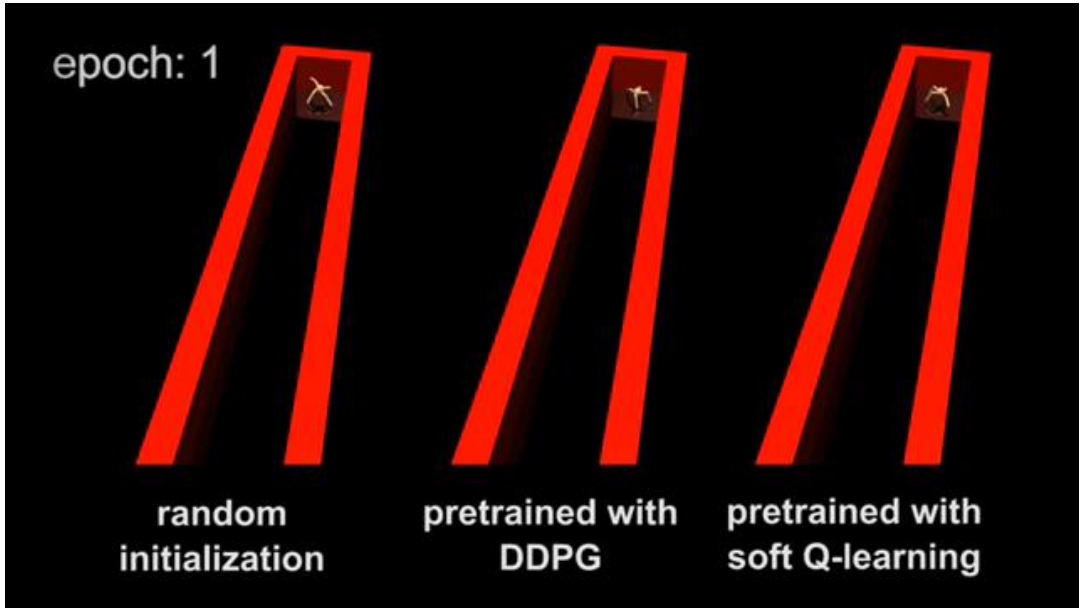
target value: $V(\mathbf{s}') = \max_{\mathbf{a}'} Q_\theta(\mathbf{s}', \mathbf{a}')$

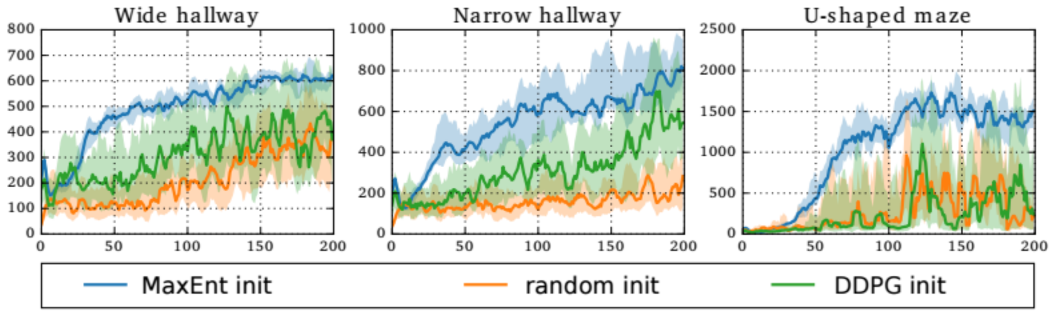
soft Q-learning: $\theta \leftarrow \theta + \alpha \nabla_\theta Q_\theta(\mathbf{s}, \mathbf{a})(r(\mathbf{s}, \mathbf{a}) + \gamma V(\mathbf{s}') - Q_\theta(\mathbf{s}, \mathbf{a}))$

target value: $V(\mathbf{s}') = \text{soft}\max_{\mathbf{a}'} Q_\theta(\mathbf{s}', \mathbf{a}') = \log \int \exp(Q_\theta(\mathbf{s}', \mathbf{a}'))d\mathbf{a}'$

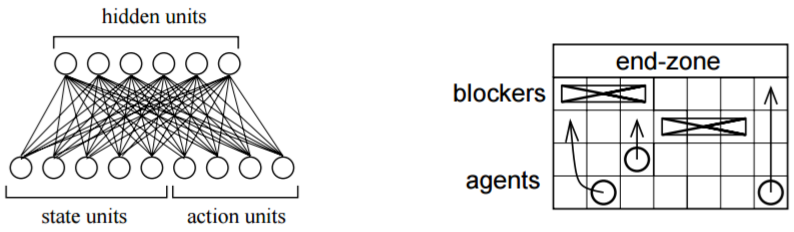
Stochastic energy-based policies provide pretraining





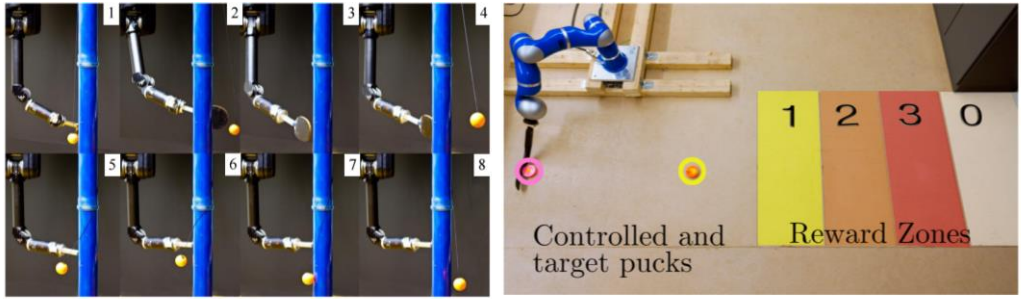


More work on maximum entropy policies



Sallans & Hinton. Using Free Energies to Represent Q-values in a Multiagent Reinforcement Learning Task. 2000.

Nachum et al. Bridging the Gap Between Value and Policy Based Reinforcement Learning. 2017.



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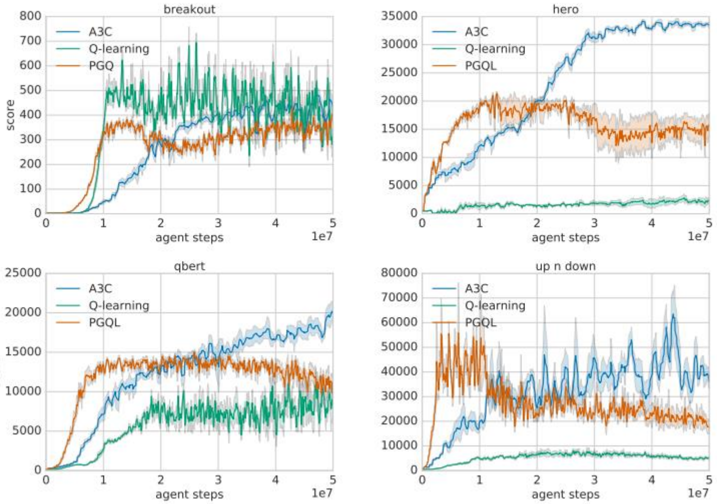

O’Donoghue et al. Combining Policy Gradient and Q-Learning. 2017

Soft optimality suggested readings

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- Todorov. (2008). General duality between optimal control and estimation: primer on the equivalence between inference and control.
- Kappen. (2009). Optimal control as a graphical model inference problem: frames control as an inference problem in a graphical model.
- Ziebart. (2010). Modeling interaction via the principle of maximal causal entropy: connection between soft optimality and maximum entropy modeling.
- Rawlik, Toussaint, Vijaykumar. (2013). On stochastic optimal control and reinforcement learning by approximate inference: temporal difference style algorithm with soft optimality.
- Haarnoja*, Tang*, Abbeel, L. (2017). Reinforcement learning with deep energy based models: soft Q-learning algorithm, deep RL with continuous actions and soft optimality
- Nachum, Norouzi, Xu, Schuurmans. (2017). Bridging the gap between value and policy based reinforcement learning.
- Schulman, Abbeel, Chen. (2017). Equivalence between policy gradients and soft Q-learning.
- Haarnoja, Zhou, Abbeel, L. (2018). Soft Actor-Critic: Off-Policy Maximum Entropy Deep Reinforcement Learning with a Stochastic Actor.
- Levine. (2018). Reinforcement Learning and Control as Probabilistic Inference: Tutorial and Review